

# $\mathcal{PT}$ -symmetric interpretation of the electromagnetic self-force

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In 1980 Englert examined the classic problem of the electromagnetic self-force on an oscillating charged particle. His approach, which was based on an earlier idea of Bateman, was to introduce a charge-conjugate particle and to show that the two-particle system is Hamiltonian. Unfortunately, Englert's model did not solve the problem of runaway modes, and the corresponding quantum theory had ghost states. It is shown here that Englert's Hamiltonian is  $\mathcal{PT}$  symmetric, and that the problems with his model arise because the  $\mathcal{PT}$  symmetry is broken at both the classical and quantum level. However, by allowing the charged particles to interact and by adjusting the coupling parameters to put the model into an unbroken  $\mathcal{PT}$ -symmetric region, one eliminates the classical runaway modes and obtains a corresponding quantum system that is ghost free.

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The techniques of  $\mathcal{PT}$  symmetry have helped to resolve a number of long-standing theoretical problems, namely, the apparent violation of unitarity in the Lee model [1], the appearance of ghosts in the Pais-Uhlenbeck model [2] and in other field-theory models [3], and the instability of the double-scaling limit in  $O(N)$  vector models [4]. In this Letter we apply these techniques to the famous old problem of runaway modes in classical electromagnetism.

An oscillating charged particle emits an electromagnetic field, and because the particle is charged, it interacts with this field. This interaction is called a *self-force*. The classical motion of the oscillating particle is described by the third-order differential equation [5]

$$m\ddot{x} + kx = m\tau\ddot{\ddot{x}}, \quad (1)$$

where  $x(t)$  represents the position of the particle as a function of time. The three-derivative term is the radiative force obtained from the Abraham-Lorentz equation. The restoring force constant of the oscillator is  $k$  and the mass of the particle is  $m$ . For an electron  $m = m_e = 9 \times 10^{-31}$  kg. The parameter  $\tau$  can be expressed as  $\tau = (4/3)r_q/c$ , where  $r_q$  is the classical radius of the charged particle; that is, the radius outside of which the electric field energy is equal to the rest-mass energy. Thus,  $\tau$  is the time for light to travel across the particle. For an electron  $\tau_e = 6 \times 10^{-24}$  sec.

The solutions to (1) suffer from the physical instabilities of runaway modes (solutions that grow exponentially with time  $t$ ) and pre-acceleration [5]. These behaviors imply that the energy of the particle is not conserved. Thus, the equation of motion (1) cannot be derived from a time-independent Hamiltonian. However, Englert [6] followed the approach that Bateman used for the damped harmonic oscillator [7] and constructed a Hamiltonian system by introducing a time-reversed version of (1):

$$m\ddot{y} + ky = -m\tau\ddot{\ddot{y}}. \quad (2)$$

It is remarkable that even though the  $x$  and  $y$  equations are separately nonconservative and are noninteracting, if they are considered together as one system, then (1) and (2) *can* be derived from a Hamiltonian. Englert interpreted the  $y$  particle as an anti- $x$  particle (that is, as a charge-conjugate version of the  $x$  particle). (In their study of the damped oscillator, Alfinito and Vitiello [8] also treated their  $y$  particle as a charge-conjugated  $x$  particle.) Unfortunately, Englert's construction failed to solve the problem of runaway solutions to the classical equations. Englert also studied the quantized version of his classical system and discovered additional problems with his model, namely, that the energy spectrum is not bounded below and that there are ghost states (states of negative norm) in the Hilbert space.

We can understand the problems that Englert encountered at the classical level (runaway modes) and at the quantum level (ghost states) if we re-examine his work from the point of view of  $\mathcal{PT}$  invariance. By appending (2), Englert created a system that is  $\mathcal{PT}$  symmetric. However, while a  $\mathcal{PT}$ -symmetric Hamiltonian system has a conserved energy, such a system can be in one of two possible states: (i) an *unbroken* state in which the classical system is in equilibrium and its frequencies are real and the corresponding quantum system has real energies and a Hilbert space with a positive inner product, or (ii) a *broken* state in which the classical system is not in equilibrium because some of its frequencies are complex and the corresponding quantum system has complex energies and a Hilbert space with ghosts. The presence of runaway modes in the classical system is a clear signal that Englert's model is in a broken  $\mathcal{PT}$ -symmetric state.

This Letter addresses the problems with Englert's Hamiltonian by allowing the  $x$  and  $y$  particles to *interact*. A typical  $\mathcal{PT}$ -symmetric system can go from a broken to an unbroken state as the coupling parameters of the system are varied [9]. Accordingly, we introduce

coupling constants that describe the interaction between the  $x$  and  $y$  particles. We then find the region of the coupling constants in which the system is in an unbroken  $\mathcal{PT}$ -symmetric state. When the system is in equilibrium, there are no runaway solutions to the equations of motion. We also present substantial evidence that the corresponding quantum system has a positive real spectrum and a Hilbert space with a positive inner product.

In physical terms, a  $\mathcal{PT}$ -symmetric system is one for which the loss and gain are balanced [10, 11]. Introducing a  $\mathcal{PT}$ -symmetric interaction between the  $x$  and  $y$  particles solves the problem of runaway modes because as one particle gains energy from the electromagnetic field, the other particle loses an equivalent amount of energy. The condition of *unbroken*  $\mathcal{PT}$  symmetry is achieved by coupling the particles sufficiently strongly so that the energy can flow fast enough from one particle to the other to maintain equilibrium.

In this Letter we construct a classical  $\mathcal{PT}$ -symmetric Hamiltonian that describes a coupled system of  $x$  and  $y$  particles and we identify the broken and unbroken regions of  $\mathcal{PT}$  symmetry; that is, the regions for which there are runaway modes and the regions for which there are no runaway modes. We then quantize the system. Our analysis of the ground state of the quantum system and our previous study of coupled damped and undamped  $\mathcal{PT}$ -symmetric oscillators [9] suggests that because the equations of motion are linear the quantum system has exactly the same parametric regions of broken and unbroken  $\mathcal{PT}$  symmetry as the corresponding classical system.

*Classical model:* Englert showed that the pair of equations (1) and (2) could be derived from the Lagrangian

$$L = -\frac{1}{2}m\tau(\dot{y}\dot{x} - \dot{y}\ddot{x}) + m\dot{x}\dot{y} - kxy. \quad (3)$$

The equations of motion (1)-(2) arise from [12]

$$\begin{aligned} 0 &= \frac{\delta L}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}}, \\ 0 &= \frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{y}}. \end{aligned} \quad (4)$$

We construct a Hamiltonian from the Lagrangian (3) by using the formula  $H = \sum_a \dot{a}p_a - L$ , where  $a = x, \dot{x}, y, \dot{y}$ :

$$H = \frac{ps - rq}{m\tau} + \frac{2rs}{m\tau^2} + \frac{pz + qw}{2} - \frac{mzw}{2} + kxy, \quad (5)$$

where we have introduced the variables  $z = \dot{x}$ ,  $w = \dot{y}$ ,  $p = p_x$ ,  $q = p_y$ ,  $r = p_z$ , and  $s = p_w$ . In this model the two systems  $x(t)$  and  $y(t)$  are noninteracting, as we can see from (1) and (2). This system is  $\mathcal{PT}$  symmetric, where the effects of parity and time reversal are given in Table I. However, it is not invariant under parity reflection  $\mathcal{P}$  or under time reversal  $\mathcal{T}$  alone.

Following the approach used in Ref. [9], we introduce

	x	y	z	w	p	q	r	s
$\mathcal{P}$	y	x	w	z	q	p	s	r
$\mathcal{T}$	x	y	-z	-w	-p	-q	r	s
$\mathcal{PT}$	y	x	-w	-z	-q	-p	s	r

TABLE I: Behaviors of the variables  $x, y, z, w, p, q, r$ , and  $s$  in the Hamiltonian (5) under space reflection  $\mathcal{P}$ , time reversal  $\mathcal{T}$ , and combined  $\mathcal{PT}$ .

two interaction terms in the equations of motion (1)-(2):

$$\begin{aligned} m\tau\ddot{x} - m\ddot{x} - kx &= Ay + B\ddot{y}, \\ m\tau\ddot{y} + m\ddot{y} + ky &= -Ax - B\ddot{x}. \end{aligned} \quad (6)$$

(There are in fact 14 possible quadratic interaction terms that we could introduce in the Hamiltonian that governs the theory and which do not increase the order of the equations of motion. The equations above are the simplest in which we observe a transition between regions of broken and unbroken  $\mathcal{PT}$  symmetry.) Figure 2 shows that at the transition from broken to unbroken  $\mathcal{PT}$  symmetry the size of the coupling parameter  $A$  is of order  $k$  and the coupling parameter  $B$  is of order  $m$ .

There are two conserved quantities for the system (6). The first quantity  $E_1$  is obtained by multiplying the first and the second equation of (6) by  $\ddot{y}$  and  $\ddot{x}$ :

$$\begin{aligned} E_1 &= m\ddot{x}\ddot{y} + k(x\ddot{y} + \ddot{x}y - \dot{x}\dot{y}) \\ &\quad + \frac{A}{2}(2x\ddot{x} + 2y\ddot{y} - \dot{x}^2 - \dot{y}^2) + \frac{B}{2}(\ddot{x}^2 + \ddot{y}^2). \end{aligned}$$

The second quantity  $E_2$  is obtained by multiplying the first and the second equation of (6) by  $\dot{y}$  and  $\dot{x}$ :

$$E_2 = m\tau(\ddot{x}\dot{y} - \dot{x}\ddot{y}) + m\dot{x}\dot{y} + kxy + \frac{A}{2}(x^2 + y^2) + \frac{B}{2}(\dot{x}^2 + \dot{y}^2).$$

The Lagrangian for the system (6) is  $\mathcal{L} = L - \frac{1}{2}A(x^2 + y^2) + \frac{1}{2}B(\dot{x}^2 + \dot{y}^2)$  and the Hamiltonian is given by

$$\mathcal{H} = H + \frac{A}{2}(x^2 + y^2) + \frac{B}{m\tau}(rw - sz). \quad (7)$$

To determine whether the system (6) is in a broken or an unbroken  $\mathcal{PT}$ -symmetric phase, we seek solutions of the form  $x(t) = \alpha e^{i\lambda t}$  and  $y(t) = \beta e^{i\lambda t}$  ( $\alpha, \beta$  constants). The frequency  $\lambda$  satisfies the sixth-degree polynomial equation

$$0 = \lambda^6 + \frac{m^2 - B^2}{m^2\tau^2}\lambda^4 + \frac{2AB - 2m}{m^2\tau^2}\lambda^2 + \frac{1 - A^2}{m^2\tau^2}. \quad (8)$$

This secular equation has real coefficients because the Hamiltonian (7) is  $\mathcal{PT}$  symmetric. The  $\mathcal{PT}$  symmetry is unbroken if we can find parameters  $(A, B)$  for which all six roots of (8) are real (so there are no exponentially

growing modes). We let  $k = 1$  without loss of generality. Then if we take  $m = 0.3$ ,  $\tau = 0.2$ , and  $A = 1.5$ , we can see from Fig. 1 that the region of unbroken symmetry is  $B > 0.425$ , the region of broken  $\mathcal{PT}$  symmetry is  $B < 0.425$ , and the  $\mathcal{PT}$  transition occurs at  $B = 0.425$ .

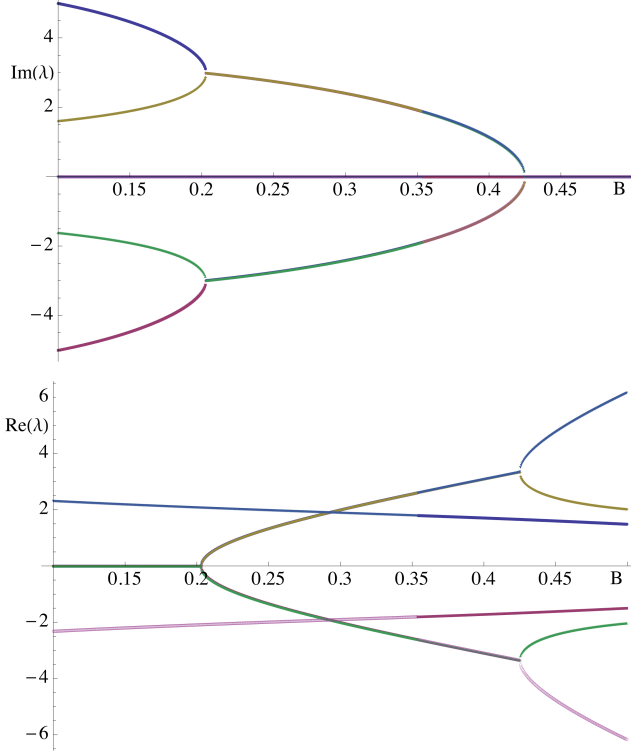


FIG. 1: Imaginary parts (upper panel) and real parts (lower panel) of the six roots  $\lambda$  of the polynomial (8) for  $m = 0.3$ ,  $\tau = 0.2$ ,  $A = 1.5$  plotted as a function of  $B$ . There is a transition from broken to unbroken  $\mathcal{PT}$  symmetry as  $B$  increases from  $B < 0.425$  to  $B > 0.425$ . The roots are real when  $B > 0.425$ .

*Quantization of the model:* The states of the quantized version of the Hamiltonian  $\mathcal{H}$  in (7) satisfy the time independent Schrödinger equation

$$\hat{\mathcal{H}}\psi_0 = E\psi_0, \quad (9)$$

where the coordinate-space Hamiltonian is

$$\begin{aligned} \hat{\mathcal{H}} = & \frac{iB(z\partial_w - w\partial_z)}{m\tau} - \frac{2\partial_{zw}^2}{m\tau^2} + \frac{\partial_{yz}^2 - \partial_{xw}^2}{m\tau} - \frac{mzw}{2} \\ & - \frac{iw\partial_y + iz\partial_x}{2} + kxy + \frac{Ax^2}{2} + \frac{Ay^2}{2}. \end{aligned} \quad (10)$$

For simplicity we restrict our attention to the ground state  $\psi_0$  only. In coordinate space this state has the form of a Gaussian in the four variables  $x$ ,  $y$ ,  $z$ , and  $w$ :

$$\begin{aligned} \psi_0 = & \exp \left\{ -\frac{m}{2} \left[ \frac{a+ib}{2\tau} x^2 + \frac{a-ib}{2\tau} y^2 + \frac{\tau(c+id)}{2} z^2 \right. \right. \\ & + \frac{\tau(c-id)}{2} w^2 + (u+iv)xz + (iv-u)yw + \frac{exy}{\tau} \\ & \left. \left. + (g+ih)xw + (ih-g)yz + n\tau zw \right] \right\}. \end{aligned} \quad (11)$$

If we substitute (11) into (9), we obtain a system of ten coupled nonlinear algebraic equations for the real coefficients  $a, b, c, d, e, g, h, n, u$ , and  $v$ :

$$0 = 2(vh - ug) + bh + eu - ag + AQ, \quad (12)$$

$$0 = u^2 + v^2 + g^2 + h^2 + au + bv - eg + Q, \quad (13)$$

$$0 = c^2 + d^2 + n^2 + 2Rd + cu + dv + ng + h + 1, \quad (14)$$

$$0 = 2(cg + nu + ug - dh - Rh) + an + b - ce, \quad (15)$$

$$0 = u^2 + v^2 + g^2 - h^2 + 2(Rv + cu + dv + ng) + ac + bd - en, \quad (16)$$

$$0 = 2cn + cg + dh + nu + v, \quad (17)$$

$$0 = 2(uh + vg) + ah + bg - ev, \quad (18)$$

$$0 = 2(ch + dg + Rg + nv + vg) - a - de + bn, \quad (19)$$

$$0 = 2(cv - du - Ru + nh + gh) + bc - ad - e, \quad (20)$$

$$0 = 2(dn + Rn) + dg - ch + nv - u, \quad (21)$$

where  $Q = 2\tau^2/m$ ,  $R = B/m$ , and the ground-state energy  $E = E_0$  is  $E_0 = (n + g)/\tau$ .

This system of equations is not easy to solve, but we have devised the following procedure to do so: First, we introduce new parameters  $X$  and  $Y$ :

$$A = (X + 1/X)/2, \quad R = (Y + 1/Y)/2. \quad (22)$$

Next, we eliminate the five variables  $c, n, g, h$ , and  $v$  by substituting

$$\begin{aligned} c &= \frac{Y^2 + 2dY + 1}{Y^2 - 1}, \quad n = -\frac{dY^2 + 2Y + d}{Y^2 - 1}, \\ g &= -\frac{u(X + Y)}{XY + 1}, \quad h = \frac{u(XY - 1)}{XY + 1}, \\ v &= \frac{u(Y - X)}{XY + 1} \end{aligned} \quad (23)$$

into all ten equations. We then observe that (17) and (21) reduce to  $0 = 0$ . Next, we solve (15) and (16) for  $a$  and  $b$  and we observe that when these variables are eliminated from the remaining equations, (19) and (20) become  $0 = 0$ . We then solve (18) for  $e$  and eliminate this variable from the other equations.

At this point, only three equations, (12), (13), and (14), remain unsolved. We solve (14) for  $d^2$  in terms of  $d$  and  $u$ . When we use this result to simplify the other two equations, we see that (12) and (13) are redundant. Finally, we solve (12) for  $d$  and eliminate  $d$  from (14). This gives a surprisingly simple fourth-degree polynomial equation for  $U = u/(XY + 1)$ :

$$\begin{aligned} 0 = & [16X^2Y(X - Y)(XY - 1)(Y^2 + 1)^2 \\ & + 8QXY^2(X + Y)^3(XY + 1)] U^4 \\ & + 8QXY(X + Y)(2X^2Y^3 - XY^4 + X - 2Y)U^3 \\ & + 2QX(Y^4 + 6Y^2 + 1)(X - Y)(XY - 1)U^2 \\ & - 2Q^2(X^2 - 1)Y^2(XY + 1)U \\ & - Q^2Y(X - Y)(XY - 1). \end{aligned} \quad (24)$$

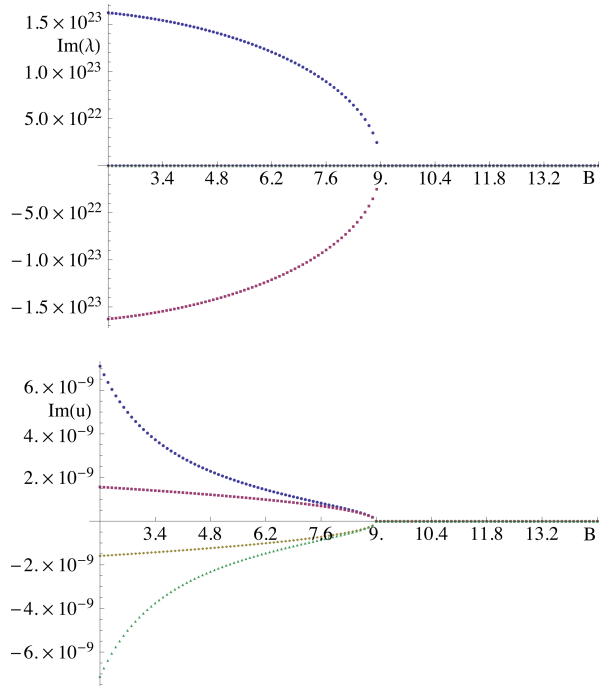


FIG. 2: Imaginary part of the roots  $\lambda$  of the classical polynomial (8) and imaginary part of the roots  $u$  of the quantum polynomial (24) for the case of a physical electron  $m = m_e$  and  $\tau = \tau_e$ . We choose  $A = 1.1$ . The parameter  $B$  is scaled by a factor  $10^{-31}$ . We can see that the  $\mathcal{PT}$  transition arises at exactly the same value of the coupling parameter  $B$  for both the classical and the quantum model; this value is approximately equal to the mass of the electron  $m_e$ .

Finally, we obtain the ground-state energy using  $E_0 = (n + g)/\tau$ . Note that there are two values of  $X$  and two values of  $Y$  for each value of  $A$  and  $R$ . Therefore, there are actually four sets of solutions to (12)-(21). Furthermore, since  $U$  satisfies a fourth-degree polynomial equation, there are a total of 16 possible values for the ground-state energy  $E_0$ . All 16 of these values are *real* when  $u$  is real, and this defines the *unbroken* region of  $\mathcal{PT}$  symmetry. In Fig. 2 we plot the imaginary part of the classical frequency  $\lambda$  and the imaginary part of  $u$  obtained from (8) and (24) for physically realistic values of the couplings  $A$  and  $B$ . Note that the  $\mathcal{PT}$  transition for the classical theory and for the quantum theory coincide.

While there are 16 possible values for  $E_0$ , only one value is physically acceptable. To show this we must calculate the energies of the higher excited states. The eigenfunctions of the  $n$ th excited states of  $\mathcal{H}$  have the form of  $\psi_0$  in (11) multiplied by a polynomial of degree  $n$  in the variables  $x$ ,  $y$ ,  $z$ , and  $w$ . We find that there is a *unique* value of  $E_0 > 0$  for which the entire spectrum of  $\mathcal{H}$  is bounded below by  $E_0$ . The details of this calculation are presented in a longer and more detailed paper [13]. Ref. [13] also describes the Stokes wedges in the complex- $x$ ,  $y$ ,  $z$ , and  $w$  planes inside of which the eigenfunctions

are normalizable and it demonstrates the orthogonality of the eigenfunctions.

We conclude with some conjectural remarks. The conventional way to explain why the classical system (1) can have runaway modes, which appear to violate the conservation of energy, is to argue that there is an infinite source of energy in the electromagnetic field of the electron. (This is because an infinite amount of work is required to assemble a pointlike electron by bringing in charge from infinity, and this work is stored in the electromagnetic field.) There are no runaway modes in the  $\mathcal{PT}$ -symmetric system (6) in the unbroken region. The underlying reason that the energy is unavailable to support the instability associated with runaway modes is that there are interference effects between the strongly coupled  $x$  and  $y$  particles in the unbroken  $\mathcal{PT}$ -symmetric region. We conjecture that one can characterize the difference between the broken and unbroken regions of  $\mathcal{PT}$ -symmetric systems by saying that the classical charge renormalization is correspondingly infinite or finite. Perhaps this distinction applies to  $\mathcal{PT}$ -symmetric quantum systems as well.

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